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## Effect of a chirp on soliton production

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We revisit the problem of influence of a chirp on production of solitons by an initial pulse, considering the rectangular pulse shape and a piecewise-constant chirp, which form of the pulse is very easy to analyze. We show that there is a critical value of the chirp, beyond which individual solitons are split. This critical value is nearly constant.

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This is a short report on the effects of a chirp on the soliton production in the nonlinear Schrödinger equation

$$iq_x = -q_{tt} - 2q^* q^2 , \tag{1}$$

which is the model equation for optical solitons in optical fibers [1]. As any pulse is injected into the end of an optical fiber, the pulse will reshape itself into a set of $N$ solitons and a collection of continuous radiation (quasilinear modes). The latter always disperses away and are of minor interest, while the solitons will propagate as coherent units down the fiber and could be counted at the other end. A major question is what are the amplitudes, positions, phases, and frequency shifts of each of the $N$ solitons. This was answered in 1972 by Zakharov and Shabat [2]. One takes the eigenvalue problem

$$v_{1t} + i\zeta v_1 = q_0 v_2 , \tag{2a}$$

$$v_{2t} - i\zeta v_2 = -q_0^* v_1 , \tag{2b}$$

where $q_0 = q(x=0,t)$ and solves for the bound state eigenvalues ($\zeta_n = \xi_n + i\eta_n$) and the normalization coefficients $c_n$ of each bound state. From $c_n$, one can determine the initial position and phase of the $n$th soliton. The one-soliton solution of (1) is

$$q = \frac{2\eta e^{i\phi}}{\cosh\theta} , \tag{3}$$

where

$$\theta = 2\eta(t - t_0) , \tag{4a}$$

$$\phi = -2\xi(t - t_0) + 4(\eta^2 + \xi^2)x + \phi_0 , \tag{4b}$$

where $\eta$ determines the amplitude and the width of the soliton, while $\xi$ determines its frequency shift ($-2\xi$).

For the case when the initial profile $q_0$ is real (no chirp) and has the shape of a box, sech, or Gaussian, the bound state spectrum has long been known [3] (see also Ref. [4]). The key feature of any simple real profile is that its area determines the number of bound states and their amplitudes ($\eta$), with the latter only mildly dependent on the shape of the initial profile. When there is a phase variation across the profile, linear in $t$, then it is a trivial matter to transform this away. The only consequence of this is to shift all eigenvalues on the real axis which introduces a common real part to all eigenvalues. The lowest order nontrivial phase variation is one that varies as $t^2$, commonly referred to as a "chirp." The effects of such a chirp on a sech profile was first studied (primarily, by numerical methods) by Hmurcik and Kaup in 1979 [5]. The basic result was that the chirp would reduce the soliton production and in some cases split a pulse into two solitons, with each soliton moving in opposite directions. Some analytical results, based on application of the WKB approximation to the linear equations (2), were obtained by Lewis [6].

In order to more carefully detail the appropriate parameter regimes, we have reinvestigated this problem. This time we have taken as a model initial profile

$$q_0 = A(t)e^{i\phi(t)} , \tag{5}$$

where $A(t)$ is a box profile, and $\phi(t)$ consists of two oppositely directed frequency shifts:

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$$A(t)=\begin{cases} Q, & t^2<L^2 \\ 0, & L^2<t^2, \end{cases} \tag{6a}$$

$$\phi(t)=k|t| \ . \tag{6b}$$

The advantage of this model is that it has piecewise closed-form solutions and, by matching the pieces together, one may obtain a closed-form solution for the scattering coefficients. The average chirp of this configuration is

$$\left\langle \frac{d^2\phi}{dt^2} \right\rangle = k/L \ . \tag{7}$$

Although the chirp in this model configuration is not continuous, it contains the key features of the problem, and the scattering data can be rapidly evaluated for it.

Typical results for the bound state eigenvalues ($\eta$ and $\xi$) are shown for $L=0$ in Figs. 1(a) ($\eta$ vs $k$ for various $Q$) and 1(b) ($\xi$ vs $k$ for various $Q$). What one observes here is that for $k$ less than about 2, one has individual soliton formation with no frequency shifts, while for $k>2$ the soliton's are created in pairs with equal and opposite phase shifts. These latter solitons will move apart and separate with a velocity growing almost linearly with the chirp.

Referring to Fig. 1(a), for $Q=1.0$, we have a net pulse area of 2.0 that is just above the first critical area of $\pi/2$, at which a soliton appears from the initial pulse at $k=0$. As $k$ increases from zero to about 2.0, the amplitude of this soliton slowly decreases and then vanishes, with no other solitons existing for larger values of $k$. For $Q=1.6$, the net initial area is 3.2, and we still have only one soliton at $k=0$. For increasing $k$, the solitons amplitude decreases, vanishing around 2.3. When $k$ takes values between about 2.8 and 12, we have various regimes where two low-amplitude solitons are produced, with equal $\eta$ and opposite $\xi$ [see Fig. 1(b)]. For larger $Q$, $Q=2.2$, one obtains solitons with a larger amplitude. For $k<2$, only one soliton is produced with a zero real part of the eigenvalue, while for $k>2$ one finds that two solitons are produced, again with exactly opposite real parts $\xi$ of the eigenvalues [Fig. 1(b)]. For $Q>2.4$, at $k=0$, the net pulse area is $>4.8$, which is above the threshold ($\frac{3}{2}\pi$) for a second soliton to be produced at $k=0$. Here, for $Q=2.8$ and 3.4, we very clearly see the existence of this second soliton. As $k$ increases, the two solitons converge and then bifurcate near $k=1.8$. For larger $k$, these two solitons now appear with nonzero real parts. Clearly what is happening is that there is a critical value of $k$. Below this critical value of $k$ ($\sim 2.0$), solitons are generated with no real part for the eigenvalue, while above this value solitons are generated in pairs, with each member of the pair moving in opposite directions.

This behavior was not obvious from the earlier work [5], which is the reason we have restudied it and presented it in this manner. One will note that there is a remarkable similarity between our Fig. 1(a) and Fig. 1 in Ref. [7], which was a seemingly different problem (vector soliton production in a birefringent fiber). However, as these figures demonstrate, their should indeed be a connection between these two problems.

In order to relate what we have done here with the earlier work of Hmurcik and Kaup [5], we also have plotted our results as they did. In Hmurcik and Kaup, what was plotted was $\eta$ vs $Q$ for various values of $k$. When we do that for these data, we obtain Fig. 2. What one observes is that for $k=1.5$, the first two eigenvalue curves start to collapse into one at the top. As $k$ increases through $k=2$, the collapse continues on down toward the bottom of the curves. Our Fig. 2 is almost identical formwise to Fig. 1 in Ref. [5].

However, what was unclear in the earlier work was the existence of a nearly uniform critical value of $k$. This is clearly seen in our Fig. 1. In Fig. 2, this is not at all obvi-

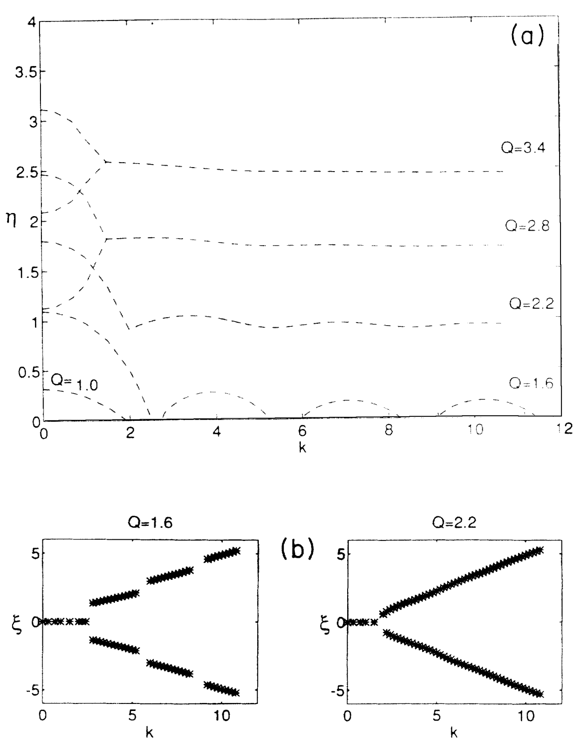


FIG. 1. Plots of the soliton eigenvalues vs the chirp factor $k$ for various $Q$. In (a), the imaginary part $\eta$ is plotted. In (b), the real part $\xi$ is plotted. (For $Q>2.2$, the curves for $\xi$ are essentially identical to the $Q=2.2$ curve).

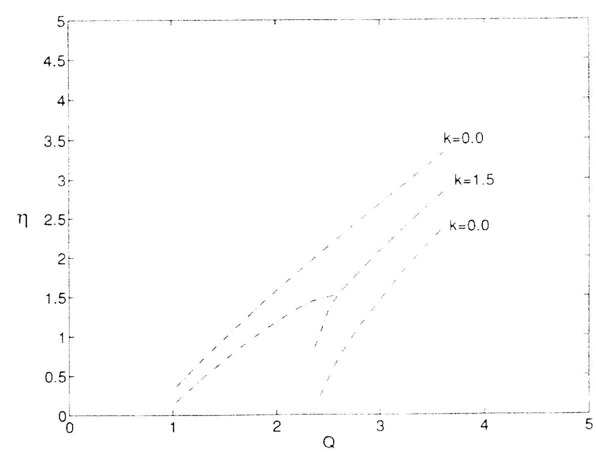


FIG. 2. Typical plot for $\eta$ vs $Q$ for $k=0$ and 1.5.

ous. Thus our result here is that for all $Q$, the critical value of $k$ seems to always be close to 2.

We note that our results presented in the bottom left-hand part of Fig. 1(a) are actually equivalent to those obtained in another early work [8] (whose authors were unaware of the earlier results of Hmurcik and Kaup [5]).

We do remark that our simplified treatment of the chirp probably does introduce one feature that may be model dependent, and may be different for a true chirp. That feature is the flatness of the curves in Fig. 1(a) for large $k$. If $\phi(t)$ in (6b) were replaced by $\phi(t)=(3k/L^3)t^2$, then different results could be expected for large $k$ for the following reason. Model (6b) has an exactly zero chirp for $t>0$ or $t<0$. Thus the region of $t>0$ and the region of $t<0$ could coalesce into individual solitons independent of the other region. This is what would happen for a true chirp if $k>2$ and, simultaneously, $k$ is not too large. These two regions would independently coalesce into separate solitons provided that $k$ was not so large as to drive the coherent area in each region below $\pi/2$. However, in the present model at $k \gg 2$, these two regions still act independently and, since the chirp is exactly zero in these regions, there is nothing to reduce the coherent area of each region. Although these regions are separated by the phase discontinuity at $t=0$, there is nothing to hinder soliton production since, in each region, we have no chirping. However, if we take $\phi(t) \propto t^2$, then for a sufficiently strong chirp ($k \gg 2$) each region would become sufficiently strongly dephased that, inside each region, the effective coherent area will be reduced below the critical area. In this case, the soliton production would be hindered or prevented. One of the major observations from Ref. [5] was that for soliton formation to occur an effective coherent area $[Q(L^3\pi/3k)^{1/2}]$ had to be on the order of or greater than $\pi/2$. Thus if the chirp is smoothly extended over the entire pulse, as $k$ increases, the eigenvalue curves in Fig. 1(a) should actually decay instead of asymptotically approaching a constant value. Except for this point, these results seem to be quite representative of how a chirp affects the soliton formation.

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